Apr14: Field extensions: simple & algebraic

The quartic question on HWZ was challeging.

§ D. Recap Let K be a Red Let f(x) EK[x] irred poly FACT KEX]/(f) is a field and K -+ K[x](f) field extersin and degree 1 KCxJ/(f) : KJ = cleg fclegree is defined as dim KCxJ/Lt IF KCL field exterior and del, then K(2) as the smallest shall containing both K &d.

We say KCL simple (prinitive) if Jath with L=K(2)

2=12+13 We say KCL simple (prinitive) $2^2 = 2 + 2\pi + 3 = 5 + 2\pi + 3$ if Jath with L=K(2) $\sim t \geq 1, d, d^2 \leq tin. indep.$ because El, 52, 53, 56) are Un. info Consider Q = Q(52) $J^{3} = (12+13)(5+216)$ Q C Q (53) $= JL_{2} + 2L_{2} + 2L_{3} + 2L_{3}$ Also have OLOZIS) < Q 556 Nuegos smaller a = 512 + 413+512+ 3612 nicons smallost subfield containing 52, 13 = 11 12 +913 \rightarrow $\{1,d,d^2,d^2\}$ are lin. indep Ques: Is QUE, 13) simple ext of Q? Is $FZ \in Span \{1, 2, d^2, d^3\}$ Suggestion: Q(TZ+TS) = Q(TZ+TS) $d^{3} - 9d = (1152 + 973) - 9(12+13)$ Certainly d= Z+13 E Q(12,13) = 72 Observation: | QLFZ,F3): Q]=4 Similarly 2-162 = 13 Conclusion: Yes, it is simple. BLUS, FS) = DLUZ+F3)

Det Let KCL be a Redet. Proposition Let KCL Bellest We say del is algebraic over K Let del be algebraic over K. Then there exists a unique numic recluible polynomical flx) EK[x] if J f(x) K[X] such that f(2)=D. with flag=D. Observadui: If & is a root of fly Ex: QCC then a is also not of f(x)g(x) for any gt/(x). . TZ JI El ave dyrbrar over Q · it Calgebrail B Krost · e, TTEC are not algebraiz/B I={f(x) <K(x] | f(d=) < <K(x] (takes work to show this) Claim: I is an ideal · e, tt E R are algebrail/R True ble observation (VFET, gekCar) (x-e o x-tr) Know KCXJ is a principal ideal dorah, • ittel are algebrail/IR i.e. every ideal Julia is of the for J=(h) for http:// (but not 2)

Therefore I some flx) EKGJ - Proposition Let KCL Buldest Let del be algebraic over K. Such Hat T = (f)Then there exists a unique monic rreduible polynombal f(x) EK[x] • $f \in I \rightarrow f(\mathcal{A} = 0)$ with fla)=D. · By dividing by leading wett, car avoire of is nonic. Observadu: If & is a root of fly then a is also not of f(x)g(x) for any gtX(x). · Why is fireducible? $If not, f=f_i \cdot f_z$ Krost $D = f(d) = f(d) f_2(d)$ I={f(x) <K[x] | f(x)=0 \$ < K[x] Asome f, bol=0 Claim: I is an ideal T=(t)c(t) c TThe ble observation (VFET, gEKCS) $= 1 (f|z(f_i)) = f_z is a scalar$ Know KCXJ is a principal ideal domain · Why is funique? For any other gla) with glated i.e. every relad JCL(kJ is of the' for J=(h) for ht/kGJ then 'flg. Ble F monie, has to be unique.

: Proposition Let KCL Beld ext Let del be algebraic over K. Then there exists a unique monil rrednible polynombal f(x) EK[x] with flag=D. We call f(x) the minunal pulphonet of 2 over K ·QCC, FZ nim poly 52 over & = X²-Z · Q(FZ) ¿Q(FZ,F3) wh poly 13 over $\overline{Q}(Fz) = \chi^2 - 3$ · What ANZ+ J3 E Q(JZ, J3) Mh poly of 2 over Q

2=12+13 $a^2 = z + z \overline{16} + 3 = \overline{5} + \overline{216}$ $\lambda^{3} = 115 + 95$ Know 21, 2, 2², 2³} lin. indep. =) J certoic f(x) with f(2)=0 $J^{4} = (J^{2})^{2} = (J^{5} + Z^{5})^{2} = Z^{5} + Y^{-} b$ +2010= 49+2016 $2^{4} - 102^{2} = 49 - 50 = -1$ $\Rightarrow f(x) = x^{9} - 10x^{2} H$ is the min poly of a over Q

Ques: What is nin poly of d=Fz+B over B(Fz)? x3-9x-252 gld=0 2², 1, 1 shoul be in dep over ELLE) 12. d = 2+16 $|\lambda^{2} - 2\sqrt{2}\lambda - 1 = 0$ \rightarrow f(x)= x²-zEx-1 they win Q(2):Q) = deg of min pilly of a over Q

2=12+13 $a^2 = z + z \overline{16} + 3 = 5 + z \overline{16}$ $J^{3} = 115 + 95$ Know {1,2,2²,2³} lin. indep. =) J certric f(x) with f(2)=0 $\lambda^{4} = (\lambda^{2})^{2} = (5 + 2\sqrt{6})^{2} = 25 + 4 \cdot 6$ + 20/6 = 494 20/6 $2^{4} - 102^{2} = 49 - 50 = -1$ = 1 2 - 102 + 1 = 0 $= \int f(x) = x^{2} - 10x^{2} H$ is the min physed over Q